Int J Theor Phys (2010) 49: 2164–2171
DOI 10.1007/s10773-010-0403-5

Raising and Lowering Operators for Orbital Angular Momentum Quantum Numbers

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Received: 28 December 2009 / Accepted: 7 June 2010 / Published online: 15 June 2010

Abstract Two vector operators aimed at shifting orbital angular momentum quantum number l successfully constructed based on the primary form proposed by Prof. X.L. Ka in 2001. The lowering operators can give the lowest angular momentum quantum numbers l for a given magnetic quantum number m in spherical harmonics $|lm\rangle$; and the state with minimum angular momentum quantum number in whole set of the spherical harmonics turns out to be $|0,0\rangle$. How to use the raising and lowering operators as acting on the state $|0,0\rangle$. to generate whole set of spherical harmonics is illustrated.

Keywords Angular momentum · Raising and lowering operators · Quantum numbers

1 Introduction

Testing an idea against a vital theoretical model usually enriches the understanding of both the idea and the model. In quantum mechanics, the ladder operator technique is widely used. For instance, the action of the angular momentum ladder operator L_+ and L_- with definition $L_\pm \equiv L_x \pm iL_y$ on spherical harmonics $|lm\rangle$ raises and lowers respectively the magnetic quantum number m by one while leaving the angular momentum quantum number l unaltered. Then is there any ladder operator that shifts the values of l in the spherical harmonics $|lm\rangle$?

Looking into literature, we can find that there are indeed results relevant to the solution to this problem. In 1980, Szpikowski and Góźdź pointed out in passing in the Appendix A of their paper [10] an operator O in the interacting boson nuclear model can diminish both l and m in $|lm\rangle$ with $m = l$ as $|l,l\rangle$ to another one $|l',l'\rangle$, where the operator O is a polynomial of terms containing powers of L_+, L_- and the tensor operators $T^{(k)}$ where superscript k denotes the rank under rotational transformations. In 1994, with help of the tensor operators T, Shanker [9] showed that the raising and lowering operator $A_\pm = A_x \pm iA_y$ constructed form

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the Lenz vector operator $\mathbf{A} = (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p})/2 - \mathbf{r}$ acting on spherical hydrogen atom eigenstates $|nlm\rangle$ happens to be $A_\pm|nll\rangle = D_{ll}^\pm|n, (l \pm 1), (l \pm 1)\rangle$, where coefficients $D_{ll}^\pm$ are constants depending on l. Burkardt and Leventhal in 2004 demonstrated that the same relation $A_\pm|nll\rangle = D_{ll}^\pm|n, (l \pm 1), (l \pm 1)\rangle$ can be obtained without resorting to the tensor operator [1].

As far as our knowledge goes, the first attempt to give a direct answer to the problem is due to Prof. Ka in 2001, who performed extensive algebraic analysis between operators $\mathbf{L}, \mathbf{r}, \mathbf{p}, \mathbf{p} \times \mathbf{L}, \mathbf{r} \times \mathbf{L}$, and L^2, etc., and observed that once acting on $|lm\rangle$ two vector operators [4],

$$\mathbf{R}(l) = \frac{i}{\hbar}\mathbf{N} \times \mathbf{L} + (l+1)\mathbf{N}, \qquad \mathbf{Q}(l) = \frac{i}{\hbar}\mathbf{N} \times \mathbf{L} - l\mathbf{N}, \tag{1}$$

are good enough to meet our need, where $\mathbf{N} \equiv \mathbf{r}/r$ is the direction operator for position. Explicitly, with defining $R_\pm(l) \equiv R_x(l) \pm i R_y(l)$, and $Q_\pm(l) \equiv Q_x(l) \pm i Q_y(l)$, Ka showed that

$$\begin{aligned} R_\pm(l)|lm\rangle &= \sqrt{\frac{2l+1}{2l+3}}a(l+2, \pm m)|l+1, m \pm 1\rangle, \\ Q_\pm(l)|lm\rangle &= \sqrt{\frac{2l+1}{2l-1}}a(l, \mp m)|l-1, m \pm 1\rangle, \end{aligned} \tag{2}$$

$$\begin{aligned} R_z(l)|lm\rangle &= \sqrt{\frac{2l+1}{2l+3}}b(l+1, m)|l+1, m\rangle, \\ Q_z(l)|lm\rangle &= -\sqrt{\frac{2l+1}{2l-1}}b(l, m)|l-1, m\rangle, \end{aligned} \tag{3}$$

where

$$a(l, m) = \mp\sqrt{(l+m)(l+m-1)}, \qquad b(l, m) = \sqrt{(l+m)(l-m)}. \tag{4}$$

Unfortunately, these two vector operators (1) contain information of the state acted, i.e., the angular momentum quantum number l. So, they have to be used as an operator as well as a state. For instance, once acted on state $|l+1, m\rangle$, quantum number l must be replaced by $l+1$. Without acting on a state, commutation relations such as $[R_i(l), Q_j(l)]$, $(i, j = 1, 2, 3)$ are meaningless. In fact, we can simply eliminate the angular momentum quantum number l with following replacement in operators (1),

$$l \to \frac{\sqrt{4L^2/\hbar^2 + 1} - 1}{2}, \quad \text{or} \quad l\hbar \to \frac{\sqrt{4L^2 + \hbar^2} - \hbar}{2} \tag{5}$$

Explicitly, the vector operators we deal with in this paper take following simple forms,

$$\mathbf{R} = i\mathbf{N} \times \frac{\mathbf{L}}{\hbar} + \mathbf{N}\frac{\sqrt{4L^2/\hbar^2 + 1} + 1}{2}, \qquad \mathbf{Q} = i\mathbf{N} \times \frac{\mathbf{L}}{\hbar} - \mathbf{N}\frac{\sqrt{4L^2/\hbar^2 + 1} - 1}{2}. \tag{6}$$

Once acting on $|lm\rangle$, we can immediately find relations $\mathbf{R}|lm\rangle = \mathbf{R}(l)|lm\rangle$ and $\mathbf{Q}|lm\rangle = \mathbf{Q}(l)|lm\rangle$. However, these two vector operators had been reported before 1999, and were confirmed in the same year in the context of the nonlinear Lie algebra [6, 7]. The present paper is mainly concerned with an illustration how to use these operators to determine the quantum states with lowest quantum numbers.

This paper is organized as following. In Sect. 2, the fundamental commutation relations are presented, and in Sect. 3, the quantum states of lowest quantum numbers are determined.

In last Sect. 4 in addition to discussions we will briefly mention some interesting topics of further studies.

2 Fundamental Commutation Relations

It is accustomed to assume that square root of an operator $L \equiv \sqrt{L^2}$ is hermitian and satisfies (Dirac [2]),

$$[L^2, L] = 0, \qquad [L, \mathbf{L}] = 0. \tag{7}$$

Choosing the simultaneous eigenvalues of two operators L^2, L_z, we have,

$$L|lm\rangle = \sqrt{l(l+1)}\hbar|lm\rangle. \tag{8}$$

Note that a vector operator $\mathbf{V}$ by definition satisfies the commutation relations, (Sakurai [8])

$$[V_i, L_j] = i\hbar\varepsilon_{ijk}V_k. \tag{9}$$

Moreover if $\mathbf{V} \cdot \mathbf{L} = 0$, it can be easily proved, (Ka [4])

$$(\mathbf{V} \times \mathbf{L}) \times \mathbf{L} = i\hbar(\mathbf{V} \times \mathbf{L}) - \mathbf{V}L^2. \tag{10}$$

As a consequence of these relations (9)–(10), we have

$$[L^2, \mathbf{N} \times \mathbf{L}] = 2i\hbar(\mathbf{N} \times \mathbf{L}) \times \mathbf{L} + 2\hbar^2\mathbf{N} \times \mathbf{L} = -2i\hbar\mathbf{N}L^2. \tag{11}$$

Using relations (9) and (11), we have immediately following commutation relation,

$$[L^2, \mathbf{R}] = \hbar^2\mathbf{R}(\sqrt{4L^2/\hbar^2 + 1} + 1). \tag{12}$$

Acting of both sides of this relation on $|lm\rangle$, we find that operator $\mathbf{R}$ really shifts the angular momentum quantum number from l to $l+1$,

$$L^2\mathbf{R}|lm\rangle = \mathbf{R}L^2|lm\rangle + \hbar^2\mathbf{R}(\sqrt{4L^2/\hbar^2 + 1} + 1)|lm\rangle = (l+1)(l+2)\hbar^2\mathbf{R}|lm\rangle. \tag{13}$$

In other words,

$$\mathbf{R}|lm\rangle \propto |l+1, m'\rangle, \tag{14}$$

where magnetic quantum number m' may differ from the original one m. Similarly, we have for operator $\mathbf{Q}$,

$$[L^2, \mathbf{Q}] = -\hbar^2\mathbf{Q}(\sqrt{4L^2/\hbar^2 + 1} - 1), \tag{15}$$

and it shifts the angular quantum number from l to $l-1$,

$$\mathbf{Q}|l, m\rangle \propto |l-1, m'\rangle. \tag{16}$$

Next, in order to examine how magnetic quantum number m changes upon the action of operators $\mathbf{R}$ and $\mathbf{Q}$, we need to calculate commutation relations such as $[L_z, \mathbf{R}]$ and $[L_z, \mathbf{Q}]$. By utilization of the definition of the vector operator (9), we have

$$[L_z, R_z] = 0, \quad \text{and} \quad [L_z, Q_z] = 0, \tag{17}$$

and

$$[L_z, R_\pm] = \pm\hbar R_\pm, \quad \text{and} \quad [L_z, Q_\pm] = \pm\hbar Q_\pm. \tag{18}$$

Equations (14) and (17) show that R_z and Q_z are operators respectively raise and lower the quantum number l in $|lm\rangle$ by one while keeping the magnetic quantum number m unchanged. Equations (14), (16) and (18) show that $R_\pm$ and $Q_\pm$ are operators respectively raise and lower l in $|lm\rangle$ by one and also move the magnetic quantum number m by ± 1 respectively. The coefficients before $|l+1, m'\rangle$ (14) and $|l-1, m'\rangle$ (16) can be obtained either by means of an algebraic way (Ka [4]) or by utilization of the recursion relations between spherical harmonics (see Appendix), and the final results are,

$$\begin{aligned} R_\pm|lm\rangle &= \sqrt{\frac{2l+1}{2l+3}}a(l+2, \pm m)|l+1, m\pm 1\rangle, \\ Q_\pm|lm\rangle &= \sqrt{\frac{2l+1}{2l-1}}a(l, \mp m)|l-1, m\pm 1\rangle, \end{aligned} \tag{19}$$

$$\begin{aligned} R_z|lm\rangle &= \sqrt{\frac{2l+1}{2l+3}}b(l+1, m)|l+1, m\rangle, \\ Q_z|lm\rangle &= -\sqrt{\frac{2l+1}{2l-1}}b(l, m)|l-1, m\rangle. \end{aligned} \tag{20}$$

Note that the operators used here do not contain the quantum number l whereas those used in (2)–(3) do.

From (19)–(20), we see that once acted on state $|lm\rangle$ two pairs of operators $R_\pm$ and $L_\pm R_z$ produce the same states with a difference of coefficients, and so are two other pairs of operators $Q_\pm$ and $L_\pm Q_z$,

$$R_\pm \sim L_\pm R_z, \qquad Q_\pm \sim L_\pm Q_z. \tag{21}$$

3 Determination of Quantum States with Lowest Quantum Numbers

From (19)–(20), operators that can lower the angular momentum quantum number are $Q_j, (j = +, -, z)$. Hence there must exist kets satisfying,

$$Q_j|lm\rangle_j = 0, \quad (j = +, -, z). \tag{22}$$

In these cases, subsequent applications of the lowering operators will produce zero kets. In contrast, by acting on these kets with appropriate raising and lowering operators and multiplying by suitable normalization factors, we can produce an infinite even whole set of the kets. For instance, once we know a state $|lm\rangle_j = |lm\rangle$, another meaningful ket $|l+p, m-q\rangle$. with $p > 0, q > 0$ and $p \geq q$ can be gotten via firstly applying $(R_-)^q$ and secondly $(R_z)^{p-q}$ on the state $|lm\rangle$, i.e., $(R_z)^{p-q}(R_-)^q|lm\rangle \propto (R_z)^{p-q}|l+q, m-q\rangle \propto |l+p, m-q\rangle$. In the following, we need to solve equations in (22) and discuss the relations between solutions. We will deal with this problem in spherical polar coordinates (θ, φ) where $Y_{lm}(\theta, \varphi) = \langle\theta\varphi|lm\rangle$. Because the vector operator **R** or **Q** contain two variables (θ, φ), solutions to (22) must depend on two quantum numbers. In general, we look for solutions that are simultaneously eigenvalue of L_z.

Firstly, we solve the differential equation $Q_z|lm\rangle_z = 0$ where the operator Q_z takes the following form,

$$Q_z = \sin\theta\frac{\partial}{\partial\theta} - \cos\theta\frac{\sqrt{4L^2/\hbar^2+1}-1}{2}, \tag{23}$$

and

$$L^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right). \tag{24}$$

In spherical polar coordinates, $Q_z|lm\rangle_z = 0$ assumes the form with $\psi_z(\theta,\varphi) = \langle\theta\varphi|lm\rangle_z$,

$$\left(\frac{\sin\theta}{\cos\theta}\frac{\partial}{\partial\theta} + \frac{1}{2}\right)\psi_z(\theta,\varphi) = \frac{\sqrt{4L^2/\hbar^2+1}}{2}\psi_z(\theta,\varphi), \tag{25}$$

or,

$$\left(\frac{\sin\theta}{\cos\theta}\frac{\partial}{\partial\theta} + \frac{1}{2}\right)\left(\frac{\sin\theta}{\cos\theta}\frac{\partial}{\partial\theta} + \frac{1}{2}\right)\psi_z(\theta,\varphi) = \left(\frac{L^2}{\hbar^2} + \frac{1}{4}\right)\psi_z(\theta,\varphi). \tag{26}$$

By standard method of separation of variables and with usual requirement of single-valuedness of the state function, general solution to (26) is given by,

$$\psi_z(\theta,\varphi) = (c_{1z}(\sin\theta)^m + c_{2z}(\sin\theta)^{-m})e^{im\varphi}, \quad (m = 0, \pm1, \pm2, \ldots), \tag{27}$$

hereafter c_i $(i = 1z, 2z, 1+, 2+, 1-, 2-)$ denote integration constants. The square integrability requires that the exponent of sine function can not be negative. The final result is simply with a normalization factor c_z

$$\psi_z(\theta,\varphi) = c_z(\sin\theta)^{|m|}e^{im\varphi}. \tag{28}$$

In terms of kets, we have $|lm\rangle_z = c_z||m|, m\rangle$. Further decrease of the quantum number l leads to $||m|-1, m\rangle$ that can be identified as a zero ket. Therefore the state $||m|, m\rangle$ really has the lowest angular momentum quantum number for the given magnetic quantum number m, implying that angular momentum quantum number $l \geq |m|$, or in other words for a given $l, m = -l, -l+1, \ldots, l$.

Secondly, we deal with other two equations $Q_-|lm\rangle_- = 0$ and $Q_+|lm\rangle_+ = 0$ in (22). Recalling the relation $Q_\pm \sim L_\pm Q_z$ in (21) i.e., $L_\pm Q_z|lm\rangle_z \sim Q_\pm|lm\rangle_z = 0$, we find that one of the two independent solutions to each of these two equations can be inferred from the two solutions (28). In fact, when $m \leq 0$, $Q_-|lm\rangle_z (= ||m|-1, m-1\rangle)$ is actually a zero ket, while $m > 0$, $Q_+|lm\rangle_z (= ||m|-1, m+1\rangle)$ is also the zero ket. One can then easily verify that another independent solution to the differential equation $Q_-|lm\rangle_- = 0$ is $||m|+1, -|m|\rangle$, whereas it is $||m|+1, |m|\rangle$ for $Q_+|lm\rangle_+ = 0$. So, the final solutions are,

$$|lm\rangle_\pm = c_{1\pm}||m|, \pm|m|\rangle + c_{2\pm}||m|+1, \pm|m|\rangle, \tag{29}$$

where upper and lower signs correspond to $Q_+|lm\rangle_+ = 0$ and $Q_-|lm\rangle_- = 0$, respectively.

Three sets of solutions with their lowest angular momentum quantum numbers corresponding to three equations (22) are completely determined respectively. Among six solutions, only four as $||m|, \pm|m|\rangle$ and $||m|+1, \pm|m|\rangle$ are independent. We comment on the solution $||m|+1, \pm|m|\rangle$ in (29) that is not the state of lowest angular momentum quantum number because it can be further lowered as $Q_z||m|+1, \pm|m|\rangle \propto ||m|, \pm|m|\rangle$. Since the minimum of magnitude of the magnetic quantum number is zero, the minimum of the angular momentum quantum number is therefore zero. So, the state with minimum quantum number in whole set of the spherical harmonics is $|0,0\rangle$. From this state $|0,0\rangle$, we can create any one $|l,m\rangle (m=-l,-l+1,\ldots,l)$. Aside from coefficients, the state $|l,|m|\rangle$ can be created via $(R_z)^{l-|m|}(R_+)^{|m|}|0,0\rangle \propto (R_z)^{l-|m|}||m|,|m|\rangle \propto |l,|m|\rangle$; and the state $|l,-|m|\rangle$ can be created via $(R_z)^{l-|m|}(Q_+)^{|m|}|0,0\rangle \propto (R_z)^{l-|m|}||m|,-|m|\rangle \propto |l,-|m|\rangle$.

4 Conclusions and Some Open Questions

Developing one step further of the work of Prof. Ka [4], we present the full form of the raising and lowering vector operators **R** and **Q** that shift the orbital angular momentum quantum number in spherical harmonics $|lm\rangle$. Apparently, three lowering operators give six different states each of them with its lowest angular momentum quantum number, and only four of them are independent. Careful analysis shows that only two of them, $||m|,\pm|m|\rangle$, bear the lowest angular momentum quantum numbers for a given the magnetic quantum number of magnitude $|m|$, and the state with minimum angular momentum quantum number in whole set of spherical harmonics turns out to be $|0,0\rangle$. Starting from this state $|0,0\rangle$, we can generate the whole set of the spherical harmonics with appropriate action of the raising and lowering operators.

The new operators (6) introduced in present paper may have wider and deeper respects worthy of future explorations. Among them we mention following two problems.

1. How to construct the shifting operators that can raise and lower the *spin* angular momentum quantum numbers is an interesting topic. For spin angular momentum operator **S**, we do not have relation $\mathbf{N}\cdot\mathbf{S}=0$ (if **S** represents spin half, $\mathbf{N}\cdot\mathbf{S}=\begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta\end{pmatrix}\neq 0$), so the relations after (10) have to be all revisited carefully and generalized.
2. An important issue is the possible connection between the operators (6) and the coherent states defined on the sphere. As reviewed by Hall and Mitchell [3], there are different forms of coherent states proposed by substantially different points of view. Among these coherent states that are defined by the eigenfunctions of annihilation or lowering operators, one is introduced by Kowalski and Rembielinski who presented the normalized vector operators [5],

$$\mathbf{Z}\equiv i\sqrt{e}\frac{2\sinh(\sqrt{4L^2/\hbar^2+1}/2)}{\sqrt{4L^2/\hbar^2+1}}\mathbf{N}\times\frac{\mathbf{L}}{\hbar} + \sqrt{e}\left(\cosh\left(\sqrt{4L^2/\hbar^2+1}/2\right)-\frac{\sinh(\sqrt{4L^2/\hbar^2+1}/2)}{\sqrt{4L^2/\hbar^2+1}}\right)\mathbf{N}. \tag{30}$$

This operator is identical to neither **R** nor **Q** but their combination with some coefficients depending on operator $\sqrt{4L^2/\hbar^2+1}$. For instance one can prove that

$$Z_z\equiv R_z\exp\left(-\frac{\sqrt{4L^2/\hbar^2+1}+1}{2}\right)\frac{1}{\sqrt{4L^2/\hbar^2+1}} + Q_z\exp\left(\frac{\sqrt{4L^2/\hbar^2+1}-1}{2}\right)\frac{1}{\sqrt{4L^2/\hbar^2+1}}. \tag{31}$$

What we ascertain now is that operators **Z**, **R** and **Q** are formed by combination of $\mathbf{N}\times\mathbf{L}$ (or $\mathbf{L}\times\mathbf{N}$) and **N** with some coefficients depending on operator $\sqrt{4L^2/\hbar^2+1}$, and there is no simple linear relation in between.

Acknowledgements This subject is supported by the National Natural Science Foundation of China. The first author is grateful to Prof. X.L. Ka, Beijing Normal University, for help discussions.

Appendix: A Derivation of the Equations (19) and (20)

The coefficients before $|l+1,m'\rangle$ (14) and $|l-1,m'\rangle$ (16) can be obtained by utilization of the recursion relations between spherical harmonics. In spherical polar coordinates, operators $R_\pm$, $Q_\pm$, R_z and Q_z take following forms,

$$R_\pm = e^{\pm i\varphi}\left(-\cos\theta\frac{\partial}{\partial\theta}\mp\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}+\sin\theta\frac{\sqrt{4L^2/\hbar^2+1}+1}{2}\right) \tag{32}$$

$$Q_\pm = e^{\pm i\varphi}\left(-\cos\theta\frac{\partial}{\partial\theta}\mp\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}-\sin\theta\frac{\sqrt{4L^2/\hbar^2+1}-1}{2}\right) \tag{33}$$

$$R_z = \sin\theta\frac{\partial}{\partial\theta}+\cos\theta\frac{\sqrt{4L^2/\hbar^2+1}+1}{2} \tag{34}$$

$$Q_z = \sin\theta\frac{\partial}{\partial\theta}-\cos\theta\frac{\sqrt{4L^2/\hbar^2+1}-1}{2} \tag{35}$$

The well-known recursion relations (Wang and Guo [11]),

$$\begin{aligned}\cos\theta Y_{lm}(\theta,\varphi) &= \sqrt{\frac{(l+m)(l-m)}{(2l+1)(2l-1)}}Y_{l-1,m}(\theta,\varphi)\\ &\quad+\sqrt{\frac{(l+m+1)(l-m+1)}{(2l+1)(2l+3)}}Y_{l+1,m}(\theta,\varphi)\end{aligned} \tag{36}$$

$$\begin{aligned}\sin\theta Y_{lm}(\theta,\varphi) &= \sqrt{\frac{(l-m)(l-m-1)}{(2l+1)(2l-1)}}Y_{l-1,m+1}(\theta,\varphi)e^{-i\varphi}\\ &\quad-\sqrt{\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}}Y_{l+1,m+1}(\theta,\varphi)e^{-i\varphi}\end{aligned} \tag{37}$$

$$\begin{aligned}\cos\theta\frac{\partial}{\partial\theta}Y_{lm}(\theta,\varphi) &= \frac{1}{2}\sqrt{(l-m)(l+m+1)}\cos\theta Y_{l,m+1}(\theta,\varphi)e^{-i\varphi}\\ &\quad-\frac{1}{2}\sqrt{(l+m)(l-m+1)}\cos\theta Y_{l,m-1}(\theta,\varphi)e^{i\varphi}\end{aligned} \tag{38}$$

$$\begin{aligned}\frac{m}{\sin\theta}Y_{lm}(\theta,\varphi) &= m\sin\theta Y_{lm}(\theta,\varphi)-\frac{1}{2}\sqrt{(l+m)(l-m+1)}\cos\theta Y_{l,m-1}(\theta,\varphi)e^{i\varphi}\\ &\quad-\frac{1}{2}\sqrt{(l-m)(l+m+1)}\cos\theta Y_{l,m+1}(\theta,\varphi)e^{-i\varphi}\end{aligned} \tag{39}$$

The action of operators (32)–(35) gives following results respectively,

$$R_{\pm}Y_{lm}(\theta,\varphi) = \mp\sqrt{\frac{(2l+1)(l\pm m+2)(l\pm m+1)}{(2l+3)}}Y_{l+1,m\pm1}(\theta,\varphi) \tag{40}$$

$$Q_{\pm}Y_{lm}(\theta,\varphi) = \mp\sqrt{\frac{(2l+1)(l\mp m)(l\mp m-1)}{(2l-1)}}Y_{l-1,m\pm1}(\theta,\varphi) \tag{41}$$

$$R_{z}Y_{lm}(\theta,\varphi) = \sqrt{\frac{(2l+1)(l+m+1)(l-m+1)}{2l+3}}Y_{l+1,m}(\theta,\varphi) \tag{42}$$

$$Q_{z}Y_{lm}(\theta,\varphi) = -\sqrt{\frac{(2l+1)(l+m)(l-m)}{2l-1}}Y_{l-1,m}(\theta,\varphi) \tag{43}$$

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